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Design of optimal sliding mode control using partial eigenstructure assignment

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This paper describes a new framework for the design of a sliding surface for a given system while multi-channel $\mathscr{H}_2$ performances of the closed-loop system are under control. In contrast to most of the current sliding surface design schemes, in this new method the level of control effort required to maintain sliding is penalised. The proposed method for the design of optimal sliding mode control is implemented in two stages. In the first stage, a state feedback gain is derived using an LMI-based scheme that can assign a number of the closed-loop eigenvalues to a known value whilst satisfying performance specifications. The sliding function matrix related to the particular state feedback derived in the first stage is obtained in the second stage by using one of the two different methods developed for this goal. The proposed theory is evaluated by using numerical examples including the problem of steady state output tracking via a state-feedback SMC for flight control.

Keywords: Multi-channel $\mathscr{H}_2$ synthesis, partial eigenstructure assignment, regional pole placement LMI characterisation, sliding surface selection.

1. Introduction

Sliding mode control (SMC) is a control method which, due to its robustness properties against matched uncertainties, has progressively been used in different applications (Argha, Li, Su, & Nguyen, 2016b; Edwards, 2004; Edwards & Spurgeon, 1998; Herrmann, Spurgeon, & Edwards, 2001; Hu, Wang, & Gao, 2008; Utkin, 1992). Roughly speaking, all the traditional SMC design methods consist of two separate stages. In the first stage, an appropriate sliding surface is chosen so that it can guarantee a reduced-order sliding motion with suitable dynamics. Many approaches have been developed for this goal; for example, pole placement and optimal quadratic (Edwards & Spurgeon, 1998), and linear matrix inequality (LMI) methods (Argha, Li, Su, & Nguyen, 2016a; Choi, 2002; Herrmann et al., 2001; Park, Choi, & Kong, 2007). Following this, the second stage designs a controller to persuade and retain the sliding motion. However, these traditional design methods are unable to limit the available control action required for satisfying the control objective. This is because, during the switching function synthesis, there is no sense of the level of the control action required to persuade and retain sliding (Edwards, 2004). If no limits are considered on

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the control actions during the design procedure, a very impractical switching surface and thereby control law may always be derived, as the high level of control efforts may be required to reach the sliding surface and maintain there thereafter.

To deal with this problem, the proposed scheme in Pan and Furuta (1994) designs the sliding surface while minimising an objective function of the system state and control input. However, as this scheme needs to ensure that at least one eigenvalue of the closed-loop system (for single input systems) is a real value, not necessarily any arbitrary weighting matrices in the objective function can result in a sliding mode control. Hence, this reference either reselects the weighting matrices or approximates the closed-loop system eigenvalues so that a set of eigenvalues are generated which can be divided into the null-space and range-space dynamics. However, no precise scheme is given on how to reselect the weighting matrices. Further, the approximation of eigenvalues may lead to a loss in optimality and possibly robustness. For addressing the limitations of Pan and Furuta (1994), Tang and Misawab (Jan, 2002) proposes an LQR-like scheme in which a weighting matrix is computed which is closest to the desired one and can result in the desired eigenvalues. Following this, the associated SMC is designed according to the obtained eigenvalues and weighting matrix. Nevertheless, both methods in Pan and Furuta (1994); Tang and Misawab (Jan, 2002) are suitable to single input systems. Alternatively, Edwards (2004) proposes two new frameworks exploiting two special system coordinate transformations, which are fundamentally different from the aforementioned schemes.

Another thread of literature has focused on the construction of the sliding surface based on manipulation of the right eigenvector and spanning the sliding subspace. For example, the main idea in Chang (2002); Chang and Chen (2000); Chen and Chang (2000); Tapia, Márquez, Bernal, and Cortez (2014) is to design a state feedback gain so that the spectrum of the obtained closed-loop system includes $n-m$ desired eigenvalues (i.e. the eigenvalues governing the reduced order sliding motion), and m (the rank of the input distribution matrix) arbitrary stable real eigenvalues. While Chang (2002); Chen and Chang (2000) consider the case where arbitrary eigenvalues are all equal, in Chang and Chen (2000) they are all different. Indeed, an appropriate choice for the set of closed-loop system eigenvalues can provide desired dynamics, optimal behaviour, and robust stability for sliding mode. However, in these schemes, it remains to investigate how to select a subset of desired eigenvalues of the overall closed-loop system by which the sliding motions are governed while minimising the performance degradation of the sliding mode dynamics compared to the overall closed-loop dynamics.

This paper aims to propose a different way for the sliding surface design in which the control effort required to induce and maintain sliding is taken into account. This approach is a middle-of-the-road method in that it uses a specific *partial eigenstructure assignment* method to assign m arbitrary stable real eigenvalues while an appropriate sliding motion dynamics will be ensured by enforcing different Lyapunov-type constraints such as the multi-channel $\mathscr{H}_2$ and regional pole placement constraints. The advantages of the proposed approach for the design of sliding surface compared to all the aforementioned references are threefold: *i*) it can set the stage for designing SMC while the level of control efforts is taken into account; *ii*) it makes it possible to integrate several Lyapunov-type constraints, e.g. regional pole placement constraints, in the SMC design problem; *iii*) the controller can be computed in a numerically very efficient method. The proposed scheme for the design of sub-optimal SMC is indeed a two-stage LMI-based approach. In the first stage, while enforcing different Lyapunov-type constraints e.g. the multi-channel $\mathscr{H}_2$, a state feedback gain is derived, using an LMI-based optimisation program employing an instrumental matrix variable, that can precisely assign some of the closed-loop eigenvalues to a priori known value. Following this, the sliding surface, associated with the state feedback gain obtained in the first stage, is determined in the second stage. Two different approaches are presented for deriving the associated switching function matrix.

Finally, it is worth mentioning that the developed $\mathscr{H}_2$-based SMC in this paper is essentially different from the $\mathscr{H}_2/\mathscr{H}_\infty$ SMC design approaches in e.g. Juma and Werner (2012); Valiloo, Khosrowjerdi, and Salari (2014), and the robust $\mathscr{H}_\infty$-based SMC with pole placement scheme in Zhang, Liu, Wang, and Karimi (2014) because in these references, during switching function synthesis, the required control effort to persuade and retain sliding is not considered. Additionally, the method presented in this manuscript is an extended version of the work in Argha, Su, Savkin, and Celler (2016).

The structure of the paper is as follows. Section 2 is dedicated to the problem statement and preliminaries. Section 3 explains the novel design strategy for the design of $\mathscr{H}_2$ based SMC. Section 4 discusses two different approaches for deriving the sliding function matrix associated with the linear controller obtained in Section 3. Section 5 summarises the proposed multichannel $\mathscr{H}_2$ based SMC. Section 6 illustrates this method via three examples including the flight control problem. Section 7 will finally conclude the paper.

Notation: herm(Σ), where Σ is a square matrix, stands for $\Sigma + \Sigma^*$ where Σ^* denotes the transpose conjugate of Σ.

2. Problem Statement and Preliminaries

Consider the following uncertain linear time invariant (LTI) continuous-time system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_2[u(t) + f(x,u,t)] \\ z(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

where $x \in \mathrm{R}^n$, $u \in \mathrm{R}^m$, and $z(t) \in \mathrm{R}^q$ are the state vector, control input vector, and $\mathscr{H}_2$ performance output vector of the system, respectively. It is also assumed that the matrices in (1) are constant and have appropriate dimensions. The unknown signal $f(x,u,t) : \mathrm{R}^n \times \mathrm{R}^m \times \mathrm{R}_+ \to \mathrm{R}^m$ denotes matched uncertainty in (1) whose Euclidean norm is bounded by a known function $\rho(x,u,t)$. Without loss of generality, it is also assumed that matrix B_2 has full rank and $m \le q \le n$. This paper aims at designing a multi-channel $\mathscr{H}_2$ based SMC for the system in (1). In doing so, this paper primarily considers the state feedback synthesis with a combination of multichannel $\mathscr{H}_2$ performance specifications. In the next section, we will develop an LMI characterisation for the multi-channel $\mathscr{H}_2$ control problems, which leads to potentially less conservative results compared to the so-called *quadratic* approach. The new LMI characterisation is very crucial for the optimal SMC of this paper, as it sets the stage for designing a certain partial eigenstructure assignment scheme which is able to assign some of the closed-loop eigenvalues to a known value.

Now, we consider a linear switching surface as:

$$\mathscr{S} = \{x : \ \sigma(t) \triangleq Sx(t) = 0\}, \tag{2}$$

where $S \in \mathrm{R}^{m\times n}$ is the full rank sliding matrix to be designed later so that the associated reduced order sliding motions have suitable dynamics.

Let us consider the following controller:

$$u(t) = -(SB_2)^{-1}(SA - \Phi S)x(t) + \vartheta(t), \tag{3}$$

where $\Phi \in \mathrm{R}^{m\times m}$ is a stable matrix, and $\vartheta(t) \in \mathrm{R}^m$ is used to denote the nonlinear part of the sliding mode controller which has the following form

$$\vartheta(t) = -(SB_2)^{-1}\rho(x,u,t)\frac{\sigma(t)}{\|\sigma(t)\|} \ \text{ if } \sigma(t) \neq 0, \tag{4}$$

in which the scalar function $\rho(\cdot)$ satisfies $\|\rho(x,u,t)\| \ge \|SB_2 f(x,u,t)\|$. We also assume that $\Phi = \lambda I_m$, where λ is a known negative scalar. As $\Phi = \lambda I_m$, the control law $u(k)$ in (3) can be reformulated as

$$u(t) = (SB_2)^{-1}SA_\lambda x(t) + \vartheta(t), \tag{5}$$

where $A_\lambda = \lambda I_n - A$. Now let $f = 0$ and $\vartheta = 0$ in (1). We then assume the controller in (5) contains only the linear part, therefore

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_2u(t) + B_1w(t) \\ z(t) &= Cx(t) + Du(t) \\ u(t) &= (SB_2)^{-1}SA_\lambda x(t), \end{aligned} \tag{6}$$

where $w(t)$ is an artificial mismatched disturbance and the distribution matrix B_1 is of appropriate dimension. The objective of this paper is to find a sliding matrix S so that the resulting reduced order motion, when restricted to $\mathscr{S}$, is stable and meets multi-channel $\mathscr{H}_2$ performance specifications. For this purpose, one may resort to solve a multi-channel $\mathscr{H}_2$ state feedback problem and thereby find the switching matrix associated with the derived optimal state feedback gain (say F). Broadly speaking, this simple scheme may not necessarily result in any solution, unless the obtained state feedback gain F can ensure that m of the closed-loop poles are exactly located at λ. In brief, in order to design a multichannel $\mathscr{H}_2$-based SMC, we need to address the following two problems:

Problem 1: *Blend the multi-channel $\mathscr{H}_2$ problem with the eigenstructure assignment method, i.e. design a state feedback F enforcing the multi-channel $\mathscr{H}_2$ constraints while ensuring m poles of the closed-loop system are precisely located at λ.*

Problem 2: *Obtain the sliding matrix S associated with the particular state feedback F, derived in Problem 1.*

The above-mentioned problems are dealt with in the following two sections.

3. Partial Egienstructure Assignment Problem For Optimal SMC Design

3.1 $\mathscr{H}_2$ LMI characterisation

Let us assume temporarily that there is no matched uncertainty in (1), i.e. $f(x,u,t) = 0$ and $\rho(x,u,t) = 0$. Now, the LMI characterisation for the (multi-channel) $\mathscr{H}_2$ problem is presented.

Lemma 1: *The following statements are equivalent:*

i) $\exists\, F$ *such that* $A + B_2F$ *is stable and* $\left\|(C+DF)\left[sI-(A+B_2F)\right]^{-1}B_1\right\|_2^2 < \gamma.$

ii) $\exists\, X > 0$ *and* $Z > 0$ *such that*

$$\begin{aligned} &\begin{bmatrix} AX + B_2Y + XA^T + Y^TB_2^T & \star \\ CX + DY & -\gamma I \end{bmatrix} < 0, \\ &\begin{bmatrix} -Z & \star \\ B_1 & -X \end{bmatrix} < 0, \\ &trace(Z) < 1, \end{aligned}$$

where $Y = FX$.

iii) $\exists\, X > 0$, $Z > 0$ *and G such that*

$$\begin{bmatrix} -(G+G^T) & \star & \star \\ AG+B_2Y+X+G & -2X & \star \\ CG+DY & 0 & -\gamma I \end{bmatrix} < 0, \tag{7}$$

$$\begin{bmatrix} -Z & \star \\ B_1 & -X \end{bmatrix} < 0, \tag{8}$$

$$trace(Z) < 1, \tag{9}$$

where $Y = FG$,

in which $X > 0$, $Z > 0$ *are s.p.d matrices, and G is a general matrix variable.*

Proof. Note that the equivalence between *i)* and *ii)* is a standard $\mathscr{H}_2$ state feedback synthesis (Boyd, Ghaoui, Feron, & Balakrishnan, 1994). Using the Schur complement, it can simply be shown that the first LMI in *iii)* can be reformulated as

$$\begin{bmatrix} -(G+G^T)+\gamma^{-1}(CG+DY)^T(CG+DY) & \star \\ AG+B_2Y+X+G & -2X \end{bmatrix} < 0.$$

Note that as $G^T + G > 0$, G is nonsingular. Performing the congruence transformation $\left[\begin{smallmatrix} G^{-T} & 0 \\ 0 & X^{-1} \end{smallmatrix}\right]$ in the above inequality leads to

$$\begin{bmatrix} -(\tilde{G}+\tilde{G}^T)+\gamma^{-1}(C^TC+F^TD^TDF) & \star \\ \tilde{X}(A+B_2F)+\tilde{X}+\tilde{G} & -2\tilde{X} \end{bmatrix} < 0.$$

where $\tilde{G} = G^{-1}$, $\tilde{X} = X^{-1}$, $F = YG^{-1}$ and $CD^T = 0$. The above inequality can be written as

$$\begin{bmatrix} \gamma^{-1}(C^TC+F^TD^TDF) & \star \\ \tilde{X}(A+B_2F)+\tilde{X} & -2\tilde{X} \end{bmatrix} + \text{herm}\left(\begin{bmatrix} -I \\ I \end{bmatrix}\tilde{G}\,[I\ 0]\right) < 0.$$

Based on the projection lemma, the above inequality holds iff the following inequalities are satisfied:

$$\begin{bmatrix} I \\ I \end{bmatrix}^T \begin{bmatrix} \gamma^{-1}(C^TC+F^TD^TDF) & \star \\ \tilde{X}(A+B_2F)+\tilde{X} & -2\tilde{X} \end{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix} < 0, \tag{10}$$

$$\begin{bmatrix} 0 \\ I \end{bmatrix}^T \begin{bmatrix} \gamma^{-1}(C^TC+F^TD^TDF) & \star \\ \tilde{X}(A+B_2F)+\tilde{X} & -2\tilde{X} \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} < 0. \tag{11}$$

As can be seen, the inequality (11) implies the trivial inequality $-\tilde{X} < 0$ and the equation (10) is

$$\tilde{X}(A+B_2F)+(A+B_2F)^T\tilde{X}+\gamma^{-1}(C^TC+F^TD^TDF) < 0.$$

Pre- and post-multiplying the above inequality by $X = \tilde{X}^{-1}$ leads to

$$AX+B_2Y+(AX+B_2Y)^T+\gamma^{-1}(XC^TCX+Y^TD^TDY) < 0,$$

where $Y = FX$. Using the Schur complement and recalling this fact that $C^TD = 0$, it is readily demonstrated that the above inequality can be written as item *ii)*. □

Remark 1:

- *It is worth noting that the dimension of the first LMI in item iii) is smaller than the one in Apkarian, Tuan, and Bernussou (2001). This makes the proposed LMI-based controller design scheme to be computationally less expensive.*
- *The specific LMI characterisation in* (7) *sets the stage for utilising different Lyapunov matrices in different LMI constraints involved in the problem. Further in* (7)*, the product terms between the system matrix A and the Lyapunov matrices* (X_i) *disappear, and the Lyapunov matrix plays no direct role in the control gain. This feature can substantially reduce the conservatism of the quadratic approach proposed for multi-objective control synthesis schemes (Chilali & Gahinet, 1996).*
- *In the robust control field, the projection lemma is usually used to eliminate the matrix variable associated with the controller state-space data. However, in the LMI-based scheme proposed for sub-optimal SMC design, we utilise the projection lemma in the opposite direction, i.e. introducing an instrumental matrix variable to the LMI-based problem.*

3.2 *Multi-channel $\mathscr{H}_2$ state feedback using improved LMI characterisations*

Now let $T_{wz}(s)$ denotes the closed-loop transfer function from w to z for control law $u = Fx$. Our target is to compute a matrix F which meets the following performance specifications

$$\begin{aligned} &\text{minimise} \quad \|T_{w_i z_i}\|_2 \\ &\text{subject to} \quad \|T_{w_1 z_1}\|_2^2 < \gamma_1, \cdots, \|T_{w_{i-1} z_{i-1}}\|_2^2 < \gamma_{i-1}, \\ &\qquad \|T_{w_{i+1} z_{i+1}}\|_2^2 < \gamma_{i+1}, \cdots, \|T_{w_{\mathscr{N}} z_{\mathscr{N}}}\|_2^2 < \gamma_{\mathscr{N}}, \end{aligned} \tag{12}$$

where $\|T_{w_i z_i}\|_2 := \|L_i T_{wz} R_i\|_2$, in which L_i and R_i are used to specify the involving channel in the associated constraint. In the sequel of this paper, we use $\mathscr{N}$ to denote the number of channels or the independent Lyapunov variables. Furthermore, a realisation of $T_{w_i z_i}$ is achieved by replacing B_1, C and D by $B_{1,i}$, C_i and D_i, $i = 1, \cdots, \mathscr{N}$, respectively, in (1). The closed-loop performance can be guaranteed by constraining (minimising) the $\mathscr{H}_2$ norm of the closed-loop transfer functions related to (input/output) signals $w_i = R_i w$ and $z_i = L_i z$; see De Oliveira, Gerome, and Bernussou (1999); Scherer, Gahinet, and Chilali (1997). Suppose that each channel is associated with a set of LMI constraints presented in (7), (8), and (9). Then the LMI characterisation for state feedback synthesis with multi-channel $\mathscr{H}_2$ specifications can be obtained by assigning a different Lyapunov variable $X_i > 0$ to every channel and exploiting common variables G and Y for all channels. As a result, using the item *iii*) of Lemma 1, the LMI characterisation for l-th channel can be represented as:

$$\begin{bmatrix} -(G+G^T) & \star & \star \\ AG + B_2 Y + X_l + G & -2X_l & \star \\ C_l G + D_l Y & 0 & -\gamma_l I \end{bmatrix} < 0, \tag{13}$$

$$\begin{bmatrix} -Z_l & \star \\ B_{1,l} & -X_l \end{bmatrix} < 0, \tag{14}$$

$$\text{trace}(Z_l) < 1, \tag{15}$$

where $X_l > 0$, $Z_l > 0$, G and Y are LMI variables, and $Y = FG$. Thereby the optimisation problem in (12) can be cast as

$$\begin{aligned} \text{minimise} \quad & \gamma_i \\ \text{subject to} \quad & (13),\ (14),\ \text{and (15) for i-th channel,} \\ & (13),\ (14),\ \text{and (15) for j-th channel} \\ & \text{with given } \gamma_j,\ j \neq i,\ j = 1, \cdots, \mathscr{N}. \end{aligned} \tag{MCH2}$$

3.3 *Partial eigenstructure assignment*

Assigning m of the closed-loop eigenvalues to a certain negative value can be performed through the LMI characterisation presented in the previous section. Indeed, the problem is to partially assign the set of eigenvalues

$$\{\overbrace{\lambda, \cdots, \lambda}^{m\ times}\}, \tag{16}$$

by state feedback. This problem can be dealt with in two steps:

1) compute the base $\begin{bmatrix} M_\lambda \\ N_\lambda \end{bmatrix}$ of nullspace of $[A - \lambda I\ B_2]$ with conformable partitioning;
2) with arbitrary $\eta_k \in \mathrm{R}^m$, $k = 1, \cdots, m$, the state feedback can be derived as $F = YG^{-1}$ with

$$Y = N\Sigma_N,\ G = M\Sigma_M, \tag{17}$$

in which

$$\begin{aligned} N &:= [\overbrace{N_\lambda, \cdots, N_\lambda}^{m\ times}, \overbrace{I, \cdots, I}^{(n-m)\ times}], \\ M &:= [\overbrace{M_\lambda, \cdots, M_\lambda}^{m\ times}, \overbrace{I, \cdots, I}^{(n-m)\ times}], \\ \Sigma_N &:= \mathrm{diag}(\eta_1, \cdots, \eta_m, \kappa_1, \cdots, \kappa_{(n-m)}), \\ \Sigma_M &:= \mathrm{diag}(\eta_1, \cdots, \eta_m, \iota_1, \cdots, \iota_{(n-m)}) \end{aligned} \tag{18}$$

with $\kappa_k \in \mathrm{R}^n$ and $\iota_k \in \mathrm{R}^n$. Note that only vectors η_k are related to the assignment of the m eigenvalues to λ. In other words, other vectors (κ_k and ι_k) are not exploited in the pole placement purposes and thereby can be employed to meet other Lyapunov-type constraints.

Now, provided by the LMI characterisation in (13), (14) and (15), the first step of our multi-channel $\mathscr{H}_2$-based SMC design can be set as an LMI program in the variables $X_i > 0$, $Z_i > 0$, $i = 1, \cdots, \mathscr{N}$, Σ_M, Σ_N and $\gamma_i > 0$, by recasting (MCH2) as:

$$\begin{aligned} \text{minimise} \quad & \gamma_i \\ \text{subject to} \quad & (13),\ (14),\ (15),\ \text{and (17) for i-th channel,} \\ & (13),\ (14),\ (15),\ \text{and (17) for j-th channel} \\ & \text{with given } \gamma_j,\ j \neq i,\ j = 1, \cdots, \mathscr{N}. \end{aligned} \tag{MHH2}$$

However, we have not yet shown that the set of closed-loop eigenvalues encompasses (16). This is the subject of the following lemma.

Lemma 2: *The set* (16) *is a subset of the closed-loop system eigenvalues, acquired by applying the state feedback $F = YG^{-1}$, with Y and G presented in* (17)*, to the system in* (1) *in the absence of uncertainty, i.e. $f = 0$.*

Proof. See Argha, Su, et al. (2016). □

Remark 2: *Rather than the proposed partial eigenstructure assignment in the first stage, it is also possible to exploit a pure pole placement method. This means that the poles that govern the sliding motion are known and will be assigned during the design procedure. The obtained result then can include those in Chen and Chang (2000); Tapia et al. (2014). However, this can limit the degrees of freedom in the problem, especially for single input systems, so that no more freedom remains for other performance constraints. Note that this eigenstructure assignment scheme is significantly different from the ones explained in Dorling and Zinober (1986); Edwards and Spurgeon (1998), in which the resulting control effort, required to persuade and retain sliding, is not taken into account.*

4. Obtaining The Switching Function Matrix

This subsection proposes two approaches to find the sliding matrix S related to the state feedback F, derived based on the partial eigenstructure assignment scheme in the previous subsection.

4.1 *Approach 1: direct approach*

The first approach is built based on the *regular form* scheme. Consider a change of coordinates $x \mapsto T_r x$. In this new coordinate system, the new matrix pair $(\tilde{A}, \tilde{B}_2)$ is of the form:

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tilde{B}_2 = \begin{bmatrix} 0 \\ B_p \end{bmatrix} \tag{19}$$

where the square matrix $B_p \in \mathrm{R}^{m\times m}$ has full rank and more importantly is invertible; see Utkin (1992). Now, the switching function matrix in the original coordinates is parameterised such that (Utkin, 1992)

$$S = S_2 \left[-\mathscr{M} \; I_m\right] T_r, \tag{20}$$

where $S_2 \in \mathrm{R}^{m\times m}$ and $\mathscr{M} = -S_2^{-1} S_1 \in \mathrm{R}^{m\times(n-m)}$ is an unknown matrix which will be derived hereafter. Note that theoretically the choice of S_2 may not influence the sliding motion (Utkin, 1992). It can readily be shown that the reduced order system matrix $A_{11} + A_{12}\mathscr{M}$ governs the sliding motion. As a result, the matrix $\mathscr{M}$ can be considered as a state feedback matrix that stabilises the reduced order matrix pair (A_{11}, A_{12}). Suppose that $\tilde{F} = [F_1 \; F_2]$ denotes the state feedback, derived based on the partial eigenstructure assignment scheme, in the new coordinate. The closed-loop system can be written as

$$\tilde{A} + \tilde{B}\tilde{F} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} + B_p F_1 & A_{22} + B_p F_2 \end{bmatrix}. \tag{21}$$

On the other hand, by applying the linear controller in (6) to the new coordinate system in (19), we have

$$\tilde{A} - \tilde{B}_2(\tilde{S}\tilde{B}_2)^{-1}\tilde{S}(\tilde{A} - \lambda I_n) = \begin{bmatrix} A_{11} & A_{12} \\ \mathscr{M}(A_{11} - \lambda I_{(n-m)}) & \mathscr{M}A_{12} + \lambda I_m \end{bmatrix}, \tag{22}$$

where $\tilde{S} := S_2 \left[-\mathscr{M} \; I_m\right]$ denotes the switching function matrix in the new coordinate. Assuming that λ does not belong to the spectrum of A_{11}, by equating the block entry $(2,1)$ of the right-hand sides of (21) and (22), one may obtain $\mathscr{M}$ as:

$$\mathscr{M} = (A_{21} + B_p F_1)(A_{11} - \lambda I_{(n-m)})^{-1}. \tag{23}$$

Now, the switching function matrix S can be derived using (20).

Remark 3: *It is also required to show that the achieved $\mathscr{M}$ can also satisfy the equality between the block entry $(2,2)$ of the right-hand sides of (21) and (22). Note that as the spectrum of the closed-loop matrix $\tilde{A} + \tilde{B}_2\tilde{F}$ includes m simple repeated λ eigenvalues, the spectrum of $\tilde{A} - \lambda I_n + \tilde{B}_2\tilde{F}$ includes m simple repeated zero eigenvalues. Hence, we may say*

$$rank\ (\tilde{A} - \lambda I_n + \tilde{B}_2\tilde{F}) = n - m.$$

Further, as $A_{11} - \lambda I_{n-m}$ is supposed to be nonsingular, we can decompose $\tilde{A} - \lambda I_n + \tilde{B}_2\tilde{F}$ as

$$\begin{bmatrix} I & 0 \\ (A_{21} + B_p F_1)(A_{11} - \lambda I_{n-m})^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} - \lambda I_{n-m} & A_{12} \\ 0 & A_{22} + B_p F_2 - (A_{21} + B_p F_1)(A_{11} - \lambda I_{n-m})^{-1} A_{12} - \lambda I_m \end{bmatrix}.$$

Since the left-hand side matrix in the equation above is full rank, the right-hand side matrix must have rank $n - m$. In other words,

$$A_{22} + B_p F_2 - (A_{21} + B_p F_1)(A_{11} - \lambda I_{n-m})^{-1} A_{12} - \lambda I_m = 0,$$

which implies the equality between the block entry $(2,2)$ of the right-hand sides of (21) and (22).

4.2 *Approach 2: indirect approach*

An alternative approach to obtain the sliding matrix is to address the equality

$$(SB_2)^{-1} SA_\lambda = F, \tag{24}$$

utilising an LMI optimisation approach as follows. As the matrix S should be such that SB_2 is invertible, let us suppose $S = B_2^T P$, with P an s.p.d matrix which will be obtained hereafter. The condition in (24) can be dealt with a simple relaxation method as:

$$\text{minimise } \mu \ \text{ subject to } \ \left\| B_2^T P(A_\lambda - B_2 F) \right\| < \mu,$$

where $\mu > 0$ is a scalar variable and F is a given state feedback matrix, obtained in the previous subsection, ensuring m of the closed-loop eigenvalues are equal to λ. Simply it can be shown that the above problem is equivalent to the following LMI minimisation problem:

$$\text{minimise } \mu \ \text{ subject to } \ \begin{bmatrix} -\mu I & \star \\ B_2^T P(A_\lambda - B_2 F) & -\mu I \end{bmatrix} < 0. \tag{25}$$

Hence, the multi-channel $\mathscr{H}_2$ based SMC problem is to find the global solution of the above minimisation problem and then the switching matrix is $S = B_2^T P$. In the case of feasibility, this problem will enforce μ to be an extremely small number associated with the precision of the computational unit.

5. The Summary of the Proposed Scheme

Now we summarise the proposed multi-channel $\mathscr{H}_2$ based SMC in the following theorem.

Theorem 1: *Assume that the optimisation problem in* (MHH2) *has a solution F for some $\gamma_i > 0$, $i = 1, \cdots \mathscr{N}$. Then the multi-channel $\mathscr{H}_2$ performance constraints $\|T_{w_i z_i}\|_2^2 < \gamma_i$, $i = 1, \cdots \mathscr{N}$ are ensured, and the resulting reduced order sliding mode dynamics, derived by the control law*

$$u(t) = Fx(t) + \vartheta(t), \tag{26}$$

where $\vartheta(t)$ is the nonlinear part of the controller introduced in (4)*, is asymptotically stable.*

Proof. Following the system coordinates in (19), suppose that $\tilde{F}$ is the state feedback gain in the new coordinate that ensures the closed-loop stability, assigns m poles of the closed-loop system at λ, and satisfies multiple $\mathscr{H}_2$ performance constraints $\|T_{w_i z_i}\|_2^2 < \gamma_i$. It can readily be shown that there exists a matrix $\mathscr{M}$ so that $\tilde{F} = (\tilde{S}\tilde{B}_2)^{-1}\tilde{S}(\tilde{A} - \lambda I_n)$, where $\tilde{S} = S_2 \begin{bmatrix} -\mathscr{M} & I_m \end{bmatrix}$. As mentioned earlier in 4.1, while $\sigma = 0$, the reduced-order sliding mode dynamics is governed by the stable reduced order system matrix $A_{11} + A_{12}\mathscr{M}$. Additionally, let us take the time derivative of (2), substitute $\dot{x}$ as the state equation (1), and use the controller (5), (4), we may obtain then

$$\dot{\sigma}(t) = \lambda\sigma(t) - \rho(x,u,t)\frac{\sigma(t)}{\|\sigma(t)\|} + SB_2 f(x,u,t). \tag{27}$$

Finally, it follows from $\|SB_2 f(x,u,t)\| \leq \rho(x,u,t)$ that the reachability condition $\frac{\sigma^T \dot{\sigma}}{\|\sigma\|} < 0$ holds. □

6. Numerical examples

The effectiveness and application of the proposed novel scheme for the design of SMC is evaluated, in this section, by the following three numerical examples.

6.1 *Example 1*

Consider the problem HE3 from COMPleib (Leibfritz & Lipinski, 2003). This problem corresponds to an eight order linearised state space model representing the dynamics of the Bell201A-1 helicopter which has 4 inputs and 6 outputs. For seeing the system matrices refer to Leibfritz and Lipinski (2003). Here in order to make a state feedback problem, we assume that all the system states are available. Also, $C = \begin{bmatrix} I_8 \\ 0_{4\times 8} \end{bmatrix}$ and $D = \begin{bmatrix} 0_{8\times 4} \\ I_4 \end{bmatrix}$.

a) The multi-objective SMC problem here is to find S such that the linear control in (5)

$$\text{minimises } \|T_{wz}\|_2 \text{ subject to } \|T_{wz}\|_\infty^2 < 1.$$

So in order to find the state-feedback F, we solve the optimisation problem in (MHH2), by replacing the LMIs (13), (14), and (15) with the following $\mathscr{H}_\infty$ LMI constraint (Shaked, Apr. 2001):

$$\begin{bmatrix} X_2 - (G + G^T) & \star & \star & \star \\ G + \nu(AG + B_2 Y) & -X_2 & \star & \star \\ CG + DY & 0 & -\nu^{-1} I & \star \\ 0 & B_1^T & 0 & -\gamma_2 \nu^{-1} I \end{bmatrix} < 0,$$

with respect to $X_2 > 0$, Y and G as decision variables, where $0 < \nu \ll 1$ is a given scalar ($\nu = 0.001$), $\lambda = -10$ and $\gamma_2 = 1$. The upper bound of the $\mathscr{H}_2$ of the closed-loop system T_{wz} is 2.1927 and the true value of $\mathscr{H}_2$ cost from w to z is 2.1603. The optimisation problem in (25) yields the sliding matrix as:

$$S = \begin{bmatrix} 176.4367 & 129.4890 & -20.9779 & -75.1873 \\ -24.4455 & -12.6388 & -7.5504 & -1.2521 \\ -65.0265 & -72.9204 & 49.6361 & 59.3065 \\ -72.9534 & -84.3384 & 109.3887 & 143.9592 \\ 60.1461 & 15.5591 & 49.4044 & 34.2311 \\ 4.0947 & 0.5397 & 5.4222 & 4.7339 \\ 186.7175 & -113.6937 & -117.7567 & -332.2428 \\ -13.3766 & -132.6635 & 353.0744 & 397.6155 \end{bmatrix}^T.$$

In such a case, the sliding motion is governed by the set of poles $\{-0.9929,\ -3.5288 \pm 0.7537i,\ -9.9707\}$.

b) For comparison, the second method in Edwards (2004) (diagonalisation procedure) is now exploited which, with the choices of $Q_q = I_n$ and $R = I_m$, gives sliding motion poles at $\{-0.2302 \pm 2.0058i - 0.5723 \pm 2.0435i\}$ and a true value $\mathscr{H}_2$ cost of 2.0950. Note that a coordinate transformation has been used to obtain the certain structure of the input matrix (B) in Edwards (2004). Besides, the min-max method explained in Edwards (2004), with $\gamma = 20$ and initial conditions $I_8/2$ and $\beta = 1$, gives sliding motion poles at $\{-0.2075 \pm 1.9876i - 0.1296 \pm 2.0099i\}$ and an $\mathscr{H}_2$ cost of 2.1096.
Solving the minimisation problem in (MCH2) with $\mathscr{N} = 1$ and additional partial eigenstructure assignment constraint in (17) (again with $\lambda = -10$), the sliding motion is governed by $\{-13.4230,\ -0.7422 \pm 2.9794i,\ -0.7117\}$ and an optimal value of 2.1632 is obtained. This is almost the same as the costs obtained from the methods of Edwards (2004). Note that for having a fair comparison, we ignored the constraints related to the second channel $\mathscr{H}_2$ performance and regional pole clustering.

c) Let us exploit the pole placement algorithm of the Matlab's Control toolbox to place the poles of the reduced order dynamics at $\{-13.4230,\ -0.7422 \pm 2.9794i,\ -0.7117\}$. The associated switching function matrix and the linear controller are obtained accordingly. The $\mathscr{H}_2$ cost, in this case, is 15.9427 which is remarkably larger than the one obtained using the proposed partial eigenstructure assignment based method.

6.2 *Example 2*

Consider now a double integrator system,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This system is considered as the first example in Edwards (2004). Let $C_1 = [10\ 5]$, $D_1 = 0.24$, $C_2 = [10\ 0]$ and $D_2 = 1$. As stated in Edwards (2004), if we solve a standard LQR problem, with the given C_2 and $D_2 = \tau$, for all choices of $\tau > 0$, to find the state feedback, the corresponding closed-loop system is governed by a complex pole pair (the damping ratio, in this case, is $\zeta = 0.7071$) Edwards (2004); Franklin, Powell, and Naeini (2002). As stated previously, this is clearly not a feasible solution to the sliding mode problem, i.e. no sliding function matrix, related to the standard LQR state feedback, exists.
Alternatively, if we solve the minimisation problem in (MHH2), with $B_1 = I$, and let $\gamma_2 = 1800$, $\lambda = -10$, we find

$$F = \begin{bmatrix} -18.8748 & -11.8875 \end{bmatrix},$$

By employing the first given approach in Section 4, the sliding function matrix is

$$S = \begin{bmatrix} 1.8875 & 1.0000 \end{bmatrix}.$$

The state feedback F causes a closed-loop system whose eigenvalues are $\{-10,\ -1.8875\}$. The true value of $\mathscr{H}_2$ cost from w to z_1 is 3.4686 and the $\mathscr{H}_2$ performance from w to z_2 is 7.5922.

For having a fair comparison, let us augment the LMI-based SMC design problem presented in Equations (27) and (28) of Edwards (2004) by the $\mathscr{H}_2$ performance constraint from w to z_2. We let $\gamma_2 = 1800$ and the decision matrix $X > 0$ be block diagonal (refer to Edwards (2004)). For convexifying the problem, we exploit a common Lyapunov decision variable in the two involved objectives. In this case, the sliding motion pole is at -5.1687 and the $\mathscr{H}_2$ performance from w to z_1 and z_2 are 5.9005 and 10.7417, respectively. The sliding function matrix is

$$S = \begin{bmatrix} 5.1687 \ 1.0000 \end{bmatrix}.$$

Figure 1 shows the trajectories of the system state by the initial condition of $x(0) = [-1 \ \ -2]^T$, and using the switching surfaces and state feedback gains obtained above. The corresponding control signals and the switching functions' evolution are also demonstrated in Figure 2. As seen, the proposed method here requires less control efforts in comparison with the diagonalisation method in Edwards (2004) for stabilising the system.

6.3 *Example 3: Flight control*

Now we consider a two-input, two-output, fourth order plant representing the motion of a Boeing B-747 aircraft obtained by linearisation around an operating condition of 20,000 ft. altitude with a speed of Mach 0.8 (Ishihara, Guo, & Takeda, 1992). The system matrices are:

$$A = \begin{bmatrix} -0.1196 & 0.0004 & -1.0001 & 0.0383 \\ -4.1195 & -0.9743 & 0.2919 & -0.0004 \\ 1.6204 & -0.0161 & -0.2320 & -0.0001 \\ 0.0007 & 1.0054 & 0.0003 & 0.0003 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.0004 & 0.0126 \\ 0.3103 & 0.1832 \\ 0.0124 & -0.9219 \\ -0.0001 & -0.0002 \end{bmatrix}, \ C_y = \begin{bmatrix} 1 \ 0 \ 0 \ 0 \\ 0 \ 0 \ 0 \ 1 \end{bmatrix},$$

and the system state, output and input vectors are

$$\begin{aligned} x(t) &= \begin{bmatrix} \beta(t) \ p(t) \ r(t) \ \phi(t) \end{bmatrix}^T, \\ y(t) &= \begin{bmatrix} \beta(t) \ \phi(t) \end{bmatrix}^T, \\ u(t) &= \begin{bmatrix} \delta_a(t) \ \delta_r(t) \end{bmatrix}^T. \end{aligned}$$

where $\beta(t)$, $p(t)$, $r(t)$, $\phi(t)$, $\delta_a(t)$ and $\delta_r(t)$ denote the sideslip angle, the roll rate, the yaw rate, the roll angle, the aileron deflection and the rudder deflection, respectively.
We provide the system with a tracking facility, by exploiting an integral action. Defining

$$\dot{\xi}(t) = \mathrm{r}(t) - y(t), \tag{28}$$

where $\mathrm{r}(t)$ is the input reference to be tracked by $y(t) = C_y x(t) \in \mathrm{R}^p$, and ξ represents the integral of the tracking error, i.e. $\mathrm{r}(t) - y(t)$, and introducing $\tilde{x} := \left[\begin{smallmatrix} \xi \\ x \end{smallmatrix}\right]$, an augmented system can be derived as:

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t) + B_{\mathrm{r}}\mathrm{r}(t), \tag{29}$$

with

$$\tilde{A} = \begin{bmatrix} 0 & -C_y \\ 0 & A \end{bmatrix}, \; \tilde{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \; B_{\mathrm{r}} = \begin{bmatrix} I_p \\ 0 \end{bmatrix}. \tag{30}$$

Note that if the matrix pair (A, B_2) is controllable and the matrix triplet (A, B_2, C_y) has no zeros at the origin, it can be shown that $(\tilde{A}, \tilde{B})$ is controllable (Alwi & Edwards, 2010). We assume the whole system states are available to the controller. Now the linear part of the control law can be considered as:

$$u(t) = \begin{bmatrix} F_{\mathrm{r}} & F \end{bmatrix} \begin{bmatrix} \xi(t) \\ x(t) \end{bmatrix} \triangleq \mathbf{F}\tilde{x}(t). \tag{31}$$

where $F \in \mathrm{R}^{m\times n}$ is the state feedback gain, $F_{\mathrm{r}} \in \mathrm{R}^{m\times p}$ is the feed-forward gain due to the reference signal $\mathrm{r}(t)$. We also let

$$C = \begin{bmatrix} \mathrm{diag}(0.1,0.1,10,10,1,1) \\ 0_{2\times 6} \end{bmatrix},$$

$$D = \begin{bmatrix} 0_{6\times 2} \\ \mathrm{diag}(1,1) \end{bmatrix},$$

$$B_1 = I_6.$$

Note that the last two nonzero terms of C is associated with the integral action and is less heavily weighted. In addition, the third and fourth terms of C have strongly been weighted in comparison with the fifth and sixth terms to provide an adequate quick closed-loop response in terms of the angular acceleration in roll and yaw. We also aim to assign the closed-loop poles in the half-plane $x < -\alpha < -0.1$.

Remark 4: *Note that the closed-loop system is said to be α stable, as stressed in Chilali and Gahinet (1996); Chilali, Gahinet, and Apkarian (1999), iff*

$$2\alpha X_{\mathscr{D}} + X_{\mathscr{D}}(A + B_2F) + (A + B_2F)^T X_{\mathscr{D}} < 0, \tag{32}$$

where $X_{\mathscr{D}} > 0$. However, the synthesis problem obtained by imposing the above α stability constraint to the synthesis problem in (MHH2) *would not be a convex problem. Alternatively, the regional pole clustering constraints can be reformulated so that the product term between the Lyapunov matrix X_i and the system matrix A is removed. It can be shown that the LMI region for an α-stability is as follows:*

$$\begin{bmatrix} -(G+G^T) & \star & \star \\ AG + B_2Y + X_i + G & -2X_i & \star \\ \sqrt{2\alpha}G & 0 & -X_i \end{bmatrix} < 0. \tag{33}$$

Now, we solve the minimisation problem in (MHH2), with $\lambda = -3$, and the solution is

$$\mathbf{F} = \begin{bmatrix} 3.3093 & 55.9821 & 13.9148 & -17.1652 & -3.6405 & -50.5282 \\ 28.8800 & 16.6514 & -20.6115 & 1.1342 & 7.2461 & -2.0027 \end{bmatrix}. \tag{34}$$

Employing the first proposed approach in Section 4, the associated sliding function matrix for the augmented system is

$$S = \begin{bmatrix} 1.7484 & 6.6057 & -0.3433 & -1.0007 & -0.0443 & -3.1216 \\ 8.9396 & 5.1543 & -4.1332 & -0.0430 & 0.9420 & 1.1509 \end{bmatrix}. \tag{35}$$

The sliding motion is governed by the set of poles $\{-2.0367 \pm 2.7934i,\ -1.6778 \pm 1.1857i\}$, and the associated true value of $\mathscr{H}_2$ cost from w to z is 25.9702. Assuming the matched uncertainty term in (1) as

$f(x,u,t) = \begin{bmatrix} 0.2\sin(t)\beta(t) \\ 0.3\sin(t)\phi(t) \end{bmatrix}$, using the proposed SMC with the obtained linear gain $\mathbf{F}$ in (34) and the associated switching function matrix S in (35), and letting the switching gain $\rho = 1$, and considering a step of 5 degrees for β during 5 to 15 s as well as a step of 2 degrees for ϕ during 30 to 40 s, Figures 3-5 show the tracking responses of the system. Note that the discontinuity in the nonlinear control term $\vartheta(t)$ in (4) is smoothed by using a sigmoidal approximation (Alwi & Edwards, 2010) as

$$\vartheta_\varepsilon(t) = -(SB_2)^{-1}\rho(x,u,t)\frac{\sigma(t)}{\varepsilon + \|\sigma(t)\|} \tag{36}$$

with the scalar $\varepsilon = 0.01$ and $\rho(x,u,t) = 1$, which this can remove the discontinuity at $\sigma = 0$ and introduce the possibility to accommodate the actuator rate limits.

6.4 *An experiment: control of a rotary pendulum system*

In this section, we consider the design of an optimal sliding mode controller for the Quanser rotary inverted pendulum system (QUBE-Servo 2 (*www.quanser.com*, 2017)). The rotary pendulum system (Furuta Pendulum) is a classic system often used in system modelling and control.

Figure 6 depicts the rotary pendulum model. In this system, the rotary arm pivot is attached to the QUBE-Servo 2 system and is actuated. The arm in this system has a length of L_r, and the moment of inertia of J_r, and if rotates counter-clockwise (CCW) its angle θ increases positively. Furthermore, if the control voltage is positive ($V_m > 0$), the servo (and thereby the arm) will turn in the CCW direction. The pendulum link, which is attached to the end of the rotary arm, has a length of L_p and its center of mass is assumed to be at $\frac{L_p}{2}$. Let us also assume that the moment of inertia about the center of mass is J_p. The angle α is zero when the inverted pendulum hang downward and increases positively when rotated CCW. The derivation of nonlinear dynamics representing this system is given in *www.quanser.com* (2017). The linear dynamics of rotary inverted pendulum are obtained by linearising the nonlinear equations about the operating point:

$$\begin{aligned} \ddot{\theta} &= \frac{1}{J_T}\left(-\left(J_p + \frac{1}{4}m_pL_p^2\right)D_r\dot{\theta} + \frac{1}{2}m_pL_pL_rD_p\dot{\alpha} + \frac{1}{4}m_p^2L_p^2L_rg\alpha + \left(J_p + \frac{1}{4}m_pL_p^2\right)\tau\right), \\ \ddot{\alpha} &= \frac{1}{J_T}\left(\frac{1}{2}m_pL_pL_rD_r\dot{\theta} - \left(J_r + m_pL_r^2\right)D_p\dot{\alpha} - \frac{1}{2}m_pL_pg(J_r + m_pL_r^2)\alpha - \frac{1}{2}m_pL_pL_r\tau\right), \end{aligned} \tag{37}$$

where

$$\begin{aligned} J_T &= J_pm_pL_r^2 + J_rJ_p + \frac{1}{4}J_rm_pL_p^2, \\ \tau &= \frac{k_m\left(V_m - k_m\dot{\theta}\right)}{R_m}. \end{aligned}$$

Here, τ denotes the torque applied at the base of the rotary arm, which is generated by the servo motor. R_m and k_m are terminal resistance and motor back-emf constant, respectively. D_r is the rotary arm viscous damping coefficient, and D_p is the pendulum damping coefficient. By substituting the values of parameters given in *www.quanser.com* (2017), the linear dynamics in (37) can be written in state space model as follows:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 149.226 & -0.0104407 & 0 \\ 0 & 261.525 & -0.0103179 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 49.7178 \\ 49.1331 \end{bmatrix} (u(t) + f(x,u,t)),$$
$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t), \tag{38}$$

where $x = [\theta,\ \alpha,\ \dot{\theta},\ \dot{\alpha}]$. In this system an observer has been provided to estimate the unmeasured system states $\dot{\theta}$ and $\dot{\alpha}$. Now we find a state feedback using *lqr* command of Matlab by employing $Q = \mathrm{diag}(10,1,1,1)$ and $R = 1$ as:

$$F_{lqr} = \begin{bmatrix} 3.1623 & -40.1000 & 1.7048 & -3.6027 \end{bmatrix}. \tag{39}$$

The closed loop poles by using F_{lqr} are located at $\{-72.7019,\ -3.1873,\ -8.1855 \pm 3.2221i\}$. Now by letting the distribution matrix $B_1 = I_4$, $C = \mathrm{diag}(\sqrt{10},1,1,1)$, $D = 1$, and $\lambda = -3.1873$ (a closed-loop real eigenvalue obtained by applying F_{lqr}), we solve the LMI-based optimisation problem in (MHH2) to obtain

$$F = \begin{bmatrix} 3.1828 & -40.2005 & 1.7134 & -3.6123 \end{bmatrix}, \tag{40}$$

and its associated sliding matrix

$$S = \begin{bmatrix} 0.3313 & -0.9370 & 0.0744 & -0.0820 \end{bmatrix}. \tag{41}$$

The closed-loop system poles are located at $\{-72.7025,\ -3.1873,\ -8.2080 \pm 3.2426i\}$. As seen, the state feedback gains obtained from *lqr* method and optimal SMC design scheme are approximately identical. The results of applying the optimal SMC using the sigmoidal approximation introduced in (36) with the scalar $\varepsilon = 0.15$ and $\rho(x,u,t) = 3.2990$ are depicted in Figure 7. Moreover, Figure 8 shows the evolution of the system state (θ) and control signal in the rotary inverted pendulum experiment using LQ regulator in (39). It is evident from these results that while the proposed optimal SMC is able to stabilise the inherently unstable rotary inverted pendulum system and effectively track the set point variations, it can remove the harmful influence of the uncertainties in the system compared to LQ regulator. Note that as the control signals in this experiment are involved with measurement noise, we have used a Savitzky-Golay smoothing filter of order 15 in the graphs showing the control signals to provide a clearer view by removing the effect of the noise. It is also necessary to mention that the larger control effort in Figure 7, compared to the one in Figure 8, is due to the nonlinear control part.

Figure 9 depicts the data acquisition system and rotary inverted pendulum exploited in this experiment.

7. Conclusions

This paper has been dedicated to the development of a novel method for the design of an SMC whose switching surface is derived from an optimisation problem constructed to meet a number of Lyapunov-type performance constraints. In doing so, in the first stage, through a convex optimisation approach, a state feedback gain is found while assigning a certain number (m) of the closed-loop system eigenvalues to a predetermined negative value, as well as satisfying multi-channel $\mathscr{H}_2$-norm constraints. Then, the proposed second stage finds the associated sliding surface. The advantages of the proposed scheme are threefold: (a) it can set the stage for designing SMC while the level of control efforts is taken into account; (b) it makes it possible to integrate a number of Lyapunov-type constraints, e.g. regional pole placement constraints, into the SMC design problem; (c) the controller can be computed in a numerically very efficient method. The results achieved from three numerical examples as well as an experiment carried out using a rotary inverted pendulum system confirm the effectiveness of the developed scheme.

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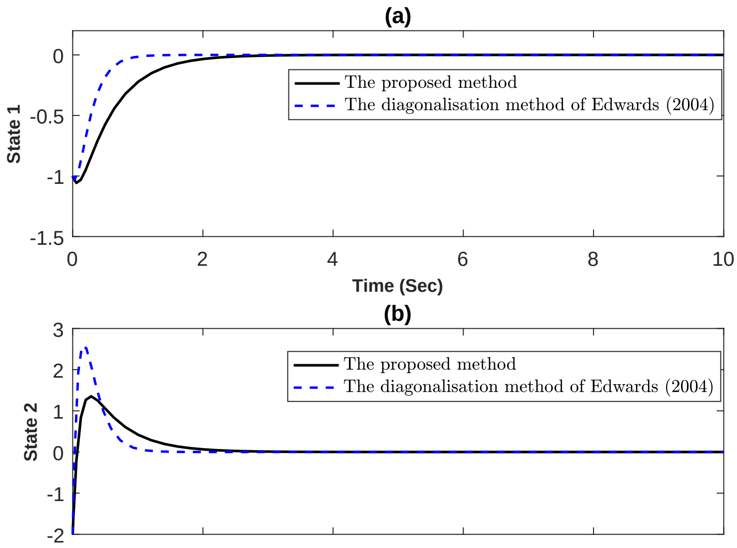

Figure 1. Evolution of the closed-loop system state trajectories in Example 2

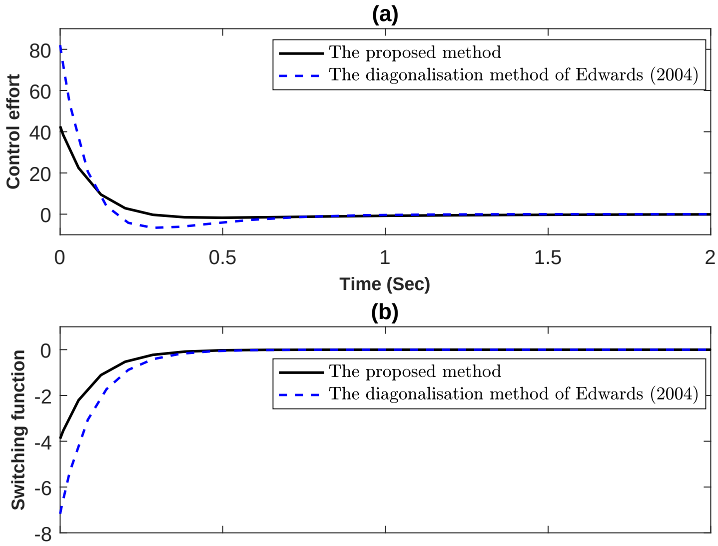

Figure 2. Control signal and evolution of the switching function in Example 2

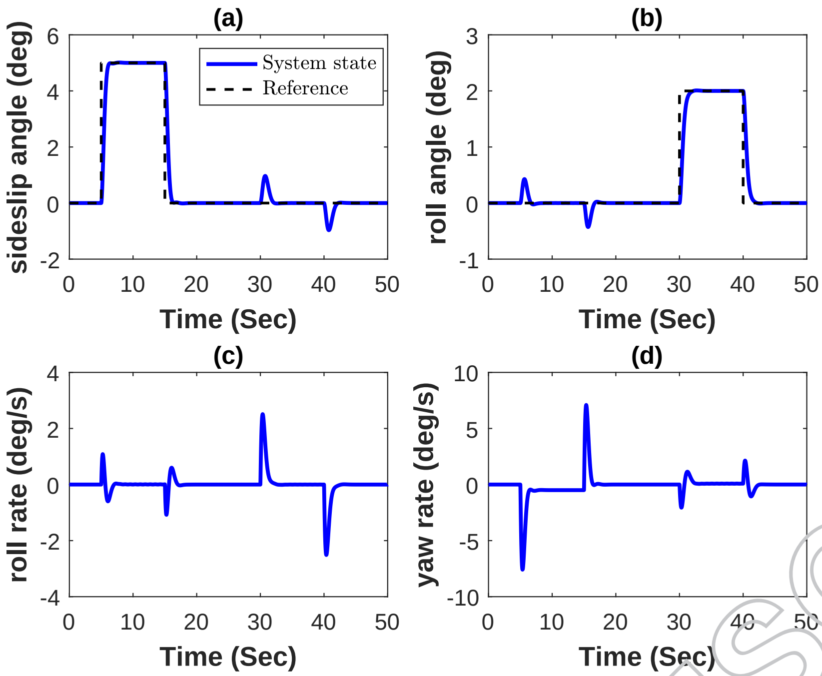

Figure 3. Evolution of the closed-loop system state trajectories in Example 3

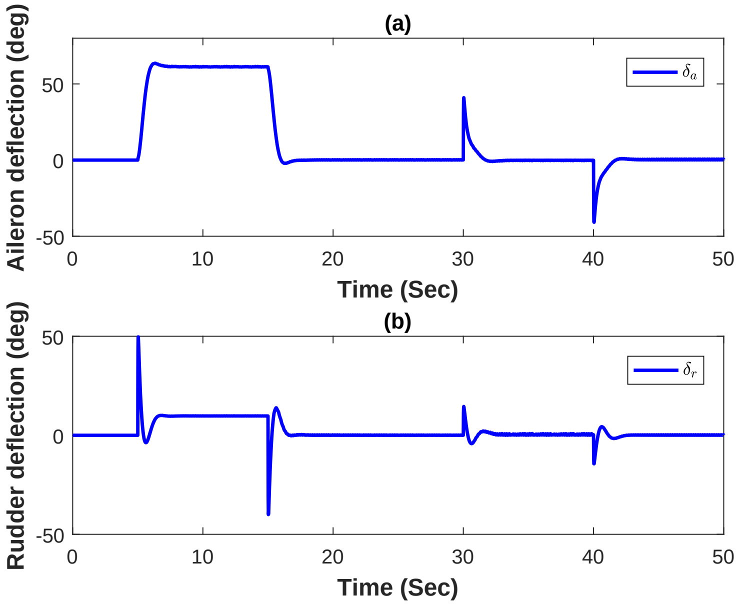

Figure 4. Control signals in Example 3

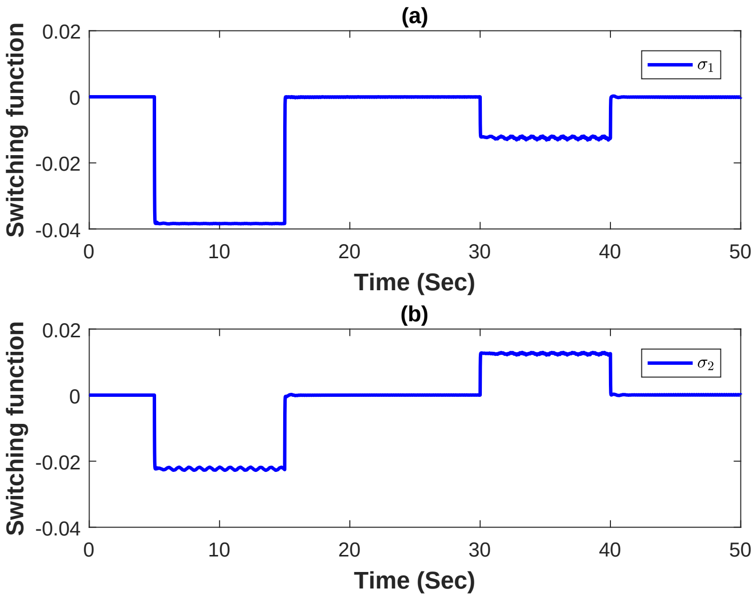

Figure 5. Evolution of the switching function in Example 3

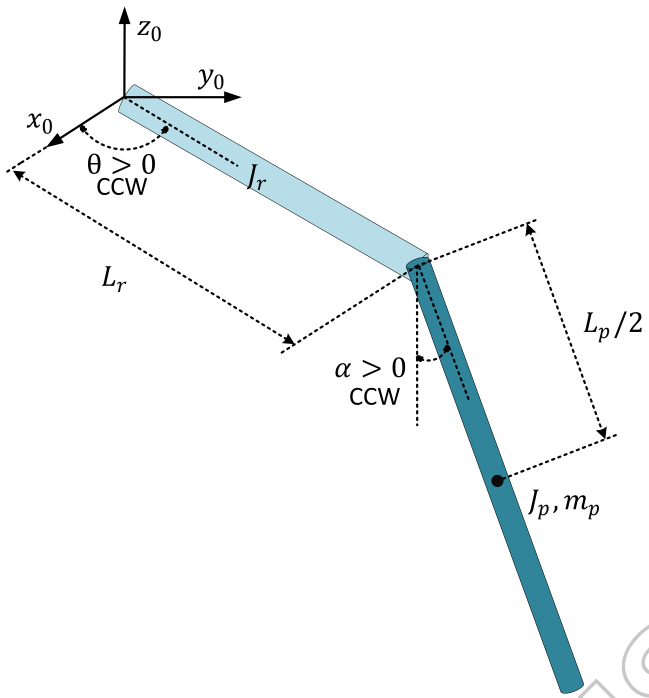

Figure 6. Rotary Inverted Pendulum Model

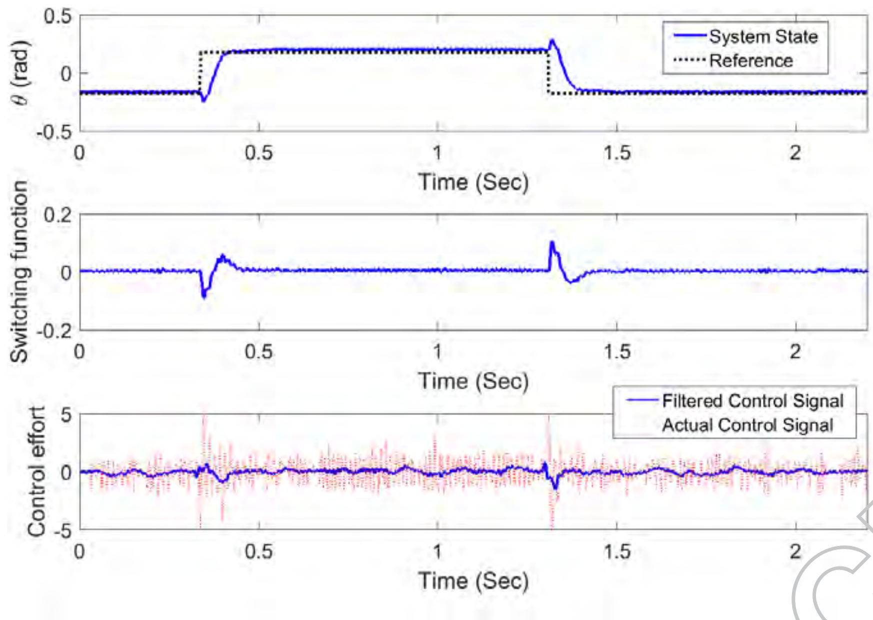

Figure 7. Evolution of the system state (θ), switching function and control signal in the rotary inverted pendulum experiment using the proposed optimal SMC

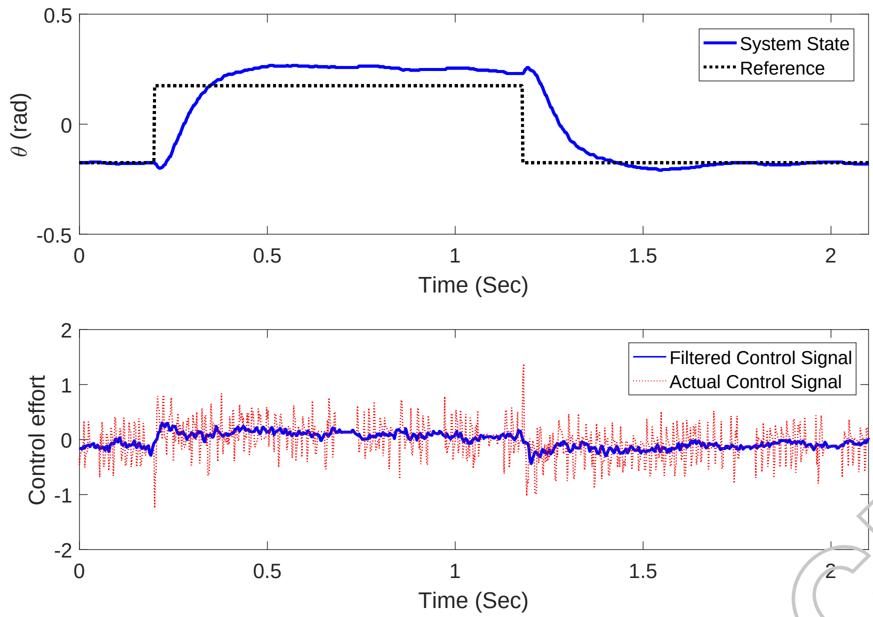

Figure 8. Evolution of the system state (θ) and control signal in the rotary inverted pendulum experiment using LQ regulator

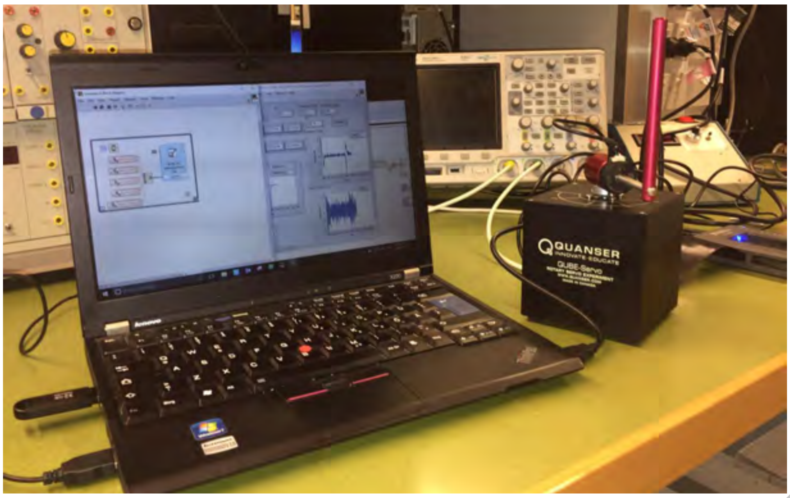

Figure 9. Rotary inverted pendulum control experiment